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1979 J. Phys. A: Math. Gen. 12 1011

(http://iopscience.iop.org/0305-4470/12/7/017)

J. Phys. A: Math. Gen., Vol. 12, No. 7, 1979. Printed in Great Britain

Complex energy eigenvalues of a zero-range atom in a uniform electric field

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Received 9 December 1977, in final form 17 November 1978

Abstract. Simple yet accurate approximate formulae for the physically interesting complex eigenvalues for a delta function well plus linear potential are presented, and their results compared with numerical results accurate to twelve significant figures. By a semiclassical analysis we suggest and confirm the correct physical interpretation of states above the top of the well, and show that contrary to widely held views, in certain potentials with isolated real singular points the higher states have a longer lifetime than lower ones. The position probability density and the phase of the wavefunction are graphically displayed, as are numerical results for dimensionless electric field intensities from 10^{-12} to beyond 10^{192}.

1. Introduction

It is well known that when an atom is placed in a uniform external electric field, its energy eigenvalues are no longer real and discrete. If a boundary condition of purely outgoing waves at large distances from the atom is imposed, the eigenvalues remain discrete but become complex. The complete determination of these Siegert eigenvalues has only been done for a few simple models of an atom in a zero field (e.g. Nussenzveig 1959, Romo 1974 etc). It is obvious that there will be differences between the properties of the quasi-stationary (or decaying, or Siegert, or resonant) states for this case and for that of a non-zero field.

Potentials containing delta functions have been used in many physical applications, whenever a short range potential must be approximated. For instance, in nuclear physics they are used to approximate short range exchange potentials; in molecular physics, the potential of each atom in a one-dimensional model of a molecular ion can be approximated by a delta function if the interatomic distance is large, i.e. in the limit of small overlap of the atomic wavefunctions.

The usual treatment of multiphoton ionisation is by means of perturbation theory where the electromagnetic field is taken as the perturbation. It is therefore not surprising to find this treatment inaccurate in the case of very strong fields. However, Peierls (1967, private communication) suggested an alternative treatment which might be more accurate in the case of very strong electric fields, i.e. where a linear potential with a simplified atomic potential is taken as the potential of the unperturbed system. The difference between the simplified and a more realistic atomic potential is then taken as a perturbation. Herrick and Stillinger (1974) have shown by a procedure of dimensionality scaling (in which the dimensionality of a problem is treated as a continuous parameter) that the strict one-dimensional analogue of a three-dimensional

Coulomb potential is a delta function, and that this property carries over to two-electron atoms.

From these considerations, it is of some interest to investigate the properties of the quasi-stationary states of the one dimensional Schrödinger equation with a linear (electric field) part together with an attractive delta function (representing an atomic nucleus) at the origin. Scheffler (1970, 1979) and Moyer (1973) both considered this case, constructed (via several different methods) the Green function, and discussed its analytic behaviour. Both have suggested that the discrete complex energy states form a complete set, but closer examination (Scheffler 1979) indicates that the picture is not quite so simple.

In § 7 we show that a semiclassical argument gives the relation between the transmission coefficient and the real and imaginary parts of the complex energy eigenvalue to high accuracy (eight significant figures or more in weak fields). Using the same semiclassical argument, we discuss a remarkable result of the exact computations: the higher the energy of a quasi-stationary state lies above the well, the longer its lifetime will be. The basic reason for this phenomenon is that for states high above the well, variations in the semiclassical period of a particle 'oscillating' between reflections at the well and the classical turning point are much stronger than variations in transmission coefficient above the well. For this as well as a multitude of non-analytic potentials (with discontinuities in the potential or in some derivative of the potential) the transmission coefficient varies slowly with energy (the reflection coefficient tending to zero only with a power of the energy), while the semiclassical period of oscillation will in general increase if, in the oscillation region, the potential varies substantially more slowly than in a harmonic oscillator. Lifetimes of states with energy high above the potential 'discontinuity' will then increase with increasing energy.

2. Eigenvalue equation

We choose s, $\mathscr{E}$ and q to denote the displacement, electric field intensity and charge in our one-dimensional model. The Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}s^2}+V\psi=E\psi, \tag{1}$$

with potential

$$V(s)=q\mathscr{E}s-\beta\delta(s), \tag{2}$$

and our further discussion can be simplified by changing to dimensionless variables ξ, $z\equiv x+iy$, ϵ, λ, F, s/l and t/τ:

$$\xi\equiv\gamma(E+q\mathscr{E}s)\equiv z+2s/(\lambda l) \tag{3}$$

$$\gamma\equiv(2m)^{1/3}(\hbar q\mathscr{E})^{-2/3}=(2/F^2)^{1/3}(\tau/\hbar). \tag{4}$$

The variable

$$z\equiv\gamma E\equiv x+\mathrm{i}y=\tfrac{1}{2}\lambda^2\epsilon=\epsilon(2/F^2)^{1/3} \tag{5}$$

is a mathematically convenient energy parameter, whereas

$$\epsilon\equiv\hbar^2E/(m\beta^2)=2z/\lambda^2=(F^2/2)^{1/3}z=\tau E/\hbar \tag{6}$$

is a parameter proportional to E, with proportionality independent of the electric field intensity $\mathscr{E}$. Likewise,

$$\lambda \equiv \beta\gamma^2 q\mathscr{E} = (4/F)^{1/3} \tag{7}$$

is a mathematically convenient parameter characterising the electric field intensity, while

$$F \equiv \hbar^4 q\mathscr{E}/(m^2\beta^3) = 4/\lambda^3 \tag{8}$$

is a dimensionless field strength proportional to $\mathscr{E}$. Also

$$l \equiv \hbar^2/(\beta m) \tag{9a}$$

is a characteristic length, and

$$\tau \equiv \hbar^3/(m\beta^2) \tag{9b}$$

a characteristic time.

Using (3) with $\delta(ax) = |a|^{-1}\delta(x)$, the Schrödinger equation (1) becomes

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} + [\xi + \lambda\delta(\xi - z)]\psi = 0. \tag{10}$$

Although the energy E and variables ξ and z will be permitted complex values, the position s and likewise the argument of the delta function will always be real.

For $\xi \neq z$, (10) reduces to the Airy equation (Abramowitz and Stegun 1964). The solution

$$u_0(\xi) \equiv \mathrm{Ai}(-\xi) \tag{11}$$

represents standing waves, and also satisfies the correct physical boundary condition on the left:

$$u_0(\xi) \to 0 \qquad \text{as} \quad s \to -\infty.$$

For $s > 0$, the solution

$$u_+(\xi) \equiv \tfrac{1}{2}(\mathrm{Ai}(-\xi) - \mathrm{i}\ \mathrm{Bi}(-\xi)) \tag{12}$$

$$= \exp(-\mathrm{i}\pi/3)\mathrm{Ai}(-\xi\exp(\mathrm{i}2\pi/3)) \tag{13}$$

$$= (12)^{-1/2}\exp(\mathrm{i}\pi/6)\xi^{1/2}H^{(1)}_{1/3}(2\xi^{3/2}/3) \tag{14}$$

represents outgoing waves at positive distances. In fact, by constructing an arbitrary wave packet from functions $U_+(\xi)$ with different energies E and using the asymptotic formula (19b) the group velocity is readily calculated to be

$$v \sim +[(2/m)(E - V(s))]^{1/2} = +[(2/m)(E + q\mathscr{E}s)]^{1/2},$$

for $\mathrm{Re}\,\xi \gg 1$.

At $\xi = z$ (i.e. $s = 0$), the wavefunction ψ is continuous, while $\psi' \equiv \mathrm{d}\psi/\mathrm{d}z$ has a discontinuity given by

$$\psi'(z^+) - \psi'(z^-) = -\lambda\psi(z). \tag{15}$$

The eigenvalue condition then easily follows:

$$u_0(z)u_+(z) = \lambda^{-1}W(u_+(z), u_0(z)) = 1/(2\pi\lambda\mathrm{i}). \tag{16}$$

The value of the Wronskian W used in the last step, and the equivalence of expressions (12)–(14), follow from Abramowitz and Stegun (1964).

3. Real energy solutions

From a physical viewpoint it is obvious that no finite *bound state* solutions can exist for finite β and $q\mathscr{E} \neq 0$, since tunnelling to $s \to \infty$ is always possible. In a potential which tends to zero as $s \to \infty$, the conditions for a real energy quasi-stationary state and for a bound state are *formally* identical, as the outgoing and bound state boundary conditions

$$\psi \sim \exp(\mathrm{i}ks) \qquad \text{and} \qquad \psi \sim \exp(-Ks)$$

are analytic continuations of each other. This is no longer true when

$$V(s) \xrightarrow{s\to\infty} -q\mathscr{E}s$$

where a bound state boundary condition is inconceivable.

We now investigate whether *real energy quasi-stationary states* are possible under any circumstances.

By assuming E to be real and taking the complex conjugate of the (real) Schrödinger equation and the real boundary condition on the left, it is clear that if ψ satisfies these, ψ^* will, too. Hence ψ and ψ^* are linearly dependent, so that ψ must be some (possibly complex) constant multiplied by a real function. It is clear that no such ψ can satisfy an outgoing (and hence complex) boundary condition on the right. Hence strictly real energy quasi-stationary states cannot occur under any circumstances.

If we now repeat our investigation for the present special case, merely by considering the eigenvalue condition (16), then with E real, $z \equiv \gamma E = x$. For real argument x, the Ai and Bi functions are real. Using (11–12), (16) becomes

$$\mathrm{Ai}^2(-x) - \mathrm{i}\,\mathrm{Ai}(-x)\mathrm{Bi}(-x) = (\mathrm{i}\pi\lambda)^{-1}.$$

The real part of this equation gives $\mathrm{Ai}(-x) = 0$, whence the imaginary part is only satisfied if $\mathrm{Bi}(-x) \to \infty$, i.e. if $x \to -\infty$; or if $\lambda \to \infty$. No finite real quasi-stationary states are possible, except apparently in the limit $\lambda \to \infty$ (i.e. when the electric field $\mathscr{E} \to 0$) which we now investigate.

4. Weak electric field

The limit

$$\lambda = (4/F)^{1/3} = (4m^2/\hbar^2 q\mathscr{E})^{1/3}\beta \to \infty \tag{17}$$

implies that either the coefficient β of the delta potential becomes large, or the electric field $\mathscr{E}$ becomes small ($F \ll 1$). Moyer (1973) showed that since

$$z = \gamma E = (\lambda^2/2)\epsilon,$$

this implies that either (a) $|z| \to \infty$ while E and ϵ remain finite, or (b) z remains finite while E and $\epsilon \to 0$. From (16) and (17) it follows that in case (b), z must approach a finite zero a_n of $u_0(z)$, or a finite zero b_n of $u_+(z)$. The zeros a_n are real and positive, whereas by (11) and (13) $b_n = a_n \exp(-2\pi\mathrm{i}/3)$. Hence we label solutions of (16) according to their behaviour for large λ (i.e. small F):

$$z_n \xrightarrow{F\to 0} a_n \overset{a_n \gg 1}{\approx} [\tfrac{3}{2}(n-\tfrac{1}{4})\pi]^{2/3} \tag{18a}$$

$$z_{-n} \xrightarrow{F\to 0} b_n = a_n \exp(-2\pi\mathrm{i}/3) \tag{18b}$$

$$z_0 = \gamma E_0 = \tfrac{1}{2}\lambda^2\epsilon_0 \xrightarrow{F\to 0} -\gamma m\beta^2/2\hbar^2 = -(\lambda/2)^2. \tag{18c}$$

In all cases, the corresponding $\epsilon_{\pm n}$ and $E_{\pm n}$ follow from (5) or (6). Case (*a*) above yields z_0, $E_0 = -m\beta^2/(2\hbar^2)$ and $\epsilon_0 = -\frac{1}{2}$ which correspond to the single bound state of a pure delta potential. The last part of (18*a*) follows from the asymptotic formula (19*a*) below.

5. Numerical results

Newton's method can be readily applied to solve the complex non-linear eigenvalue condition (16). One may, for instance, select a very small field F and use (18) to start the iteration, which converges stably and rapidly. Successively doubling F (or even multiplying by factors as large as ten) and using the previous eigenvalue z (or an extrapolate from the previous few eigenvalues) as initial guess for the present z, one can rapidly generate the results shown in figures 1 to 3. For those states z_n for which $n > 0$, an even simpler and more efficient method is to use the procedure outlined below equation (23*b*) to obtain starting values for Newton's method. In this manner, isolated eigenvalues are readily obtained. The neat symmetry of the eigenvalues in the z plane is distorted when transformed to the $\epsilon = (F^2/2)^{1/3}z$ plane because different parts of each curve correspond to different values of F. The (isonodal) curves obtained by continuously varying $F = 4/\lambda^3$ can of course not cross at any point where F has the same value on both. For the states z_n or ϵ_n, with $n \geqslant 0$, each solid curve of quasi-stationary energy

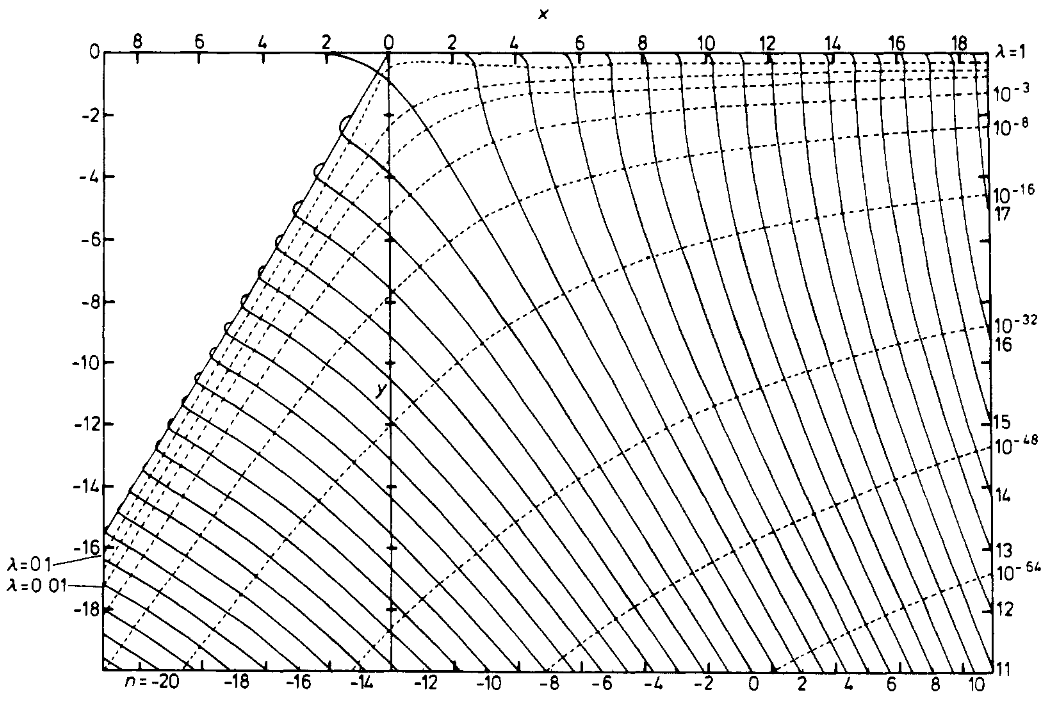

Figure 1. The complex $z = \gamma E$ plane of eigenvalues. The full curves are adiabatic invariant or isonodal curves, with nodal quantum numbers from -21 to $+17$. The intersection points with broken curves are eigenvalues for a specific potential, i.e. for given $\lambda = (4/F)^{1/3}$. The line $\arg z = -2\pi/3$ is also indicated.

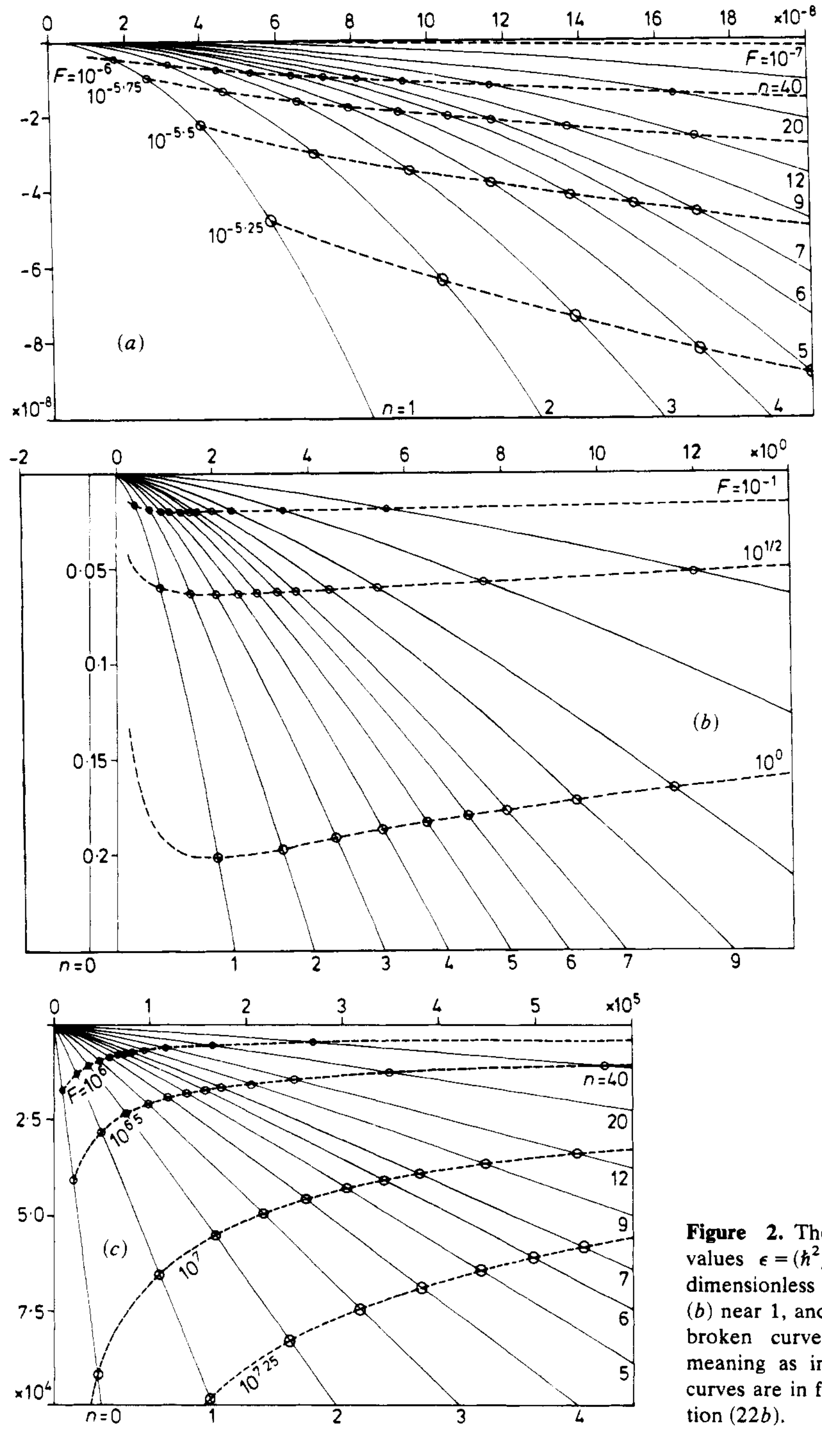

Figure 2. The complex eigenvalues $\epsilon = (\hbar^2/m\beta^2)E$ when the dimensionless field F is (a) small, (b) near 1, and (c) large. Full and broken curves have the same meaning as in figure 1. Broken curves are in fact graphs of equation (22b).

eigenvalues corresponds to a wavefunction with n quasi-nodes (see figure 3), and may accordingly be called an isonodal curve. Or since each curve is obtained by continuously varying a parameter F in the potential, it may be termed an adiabatic invariant

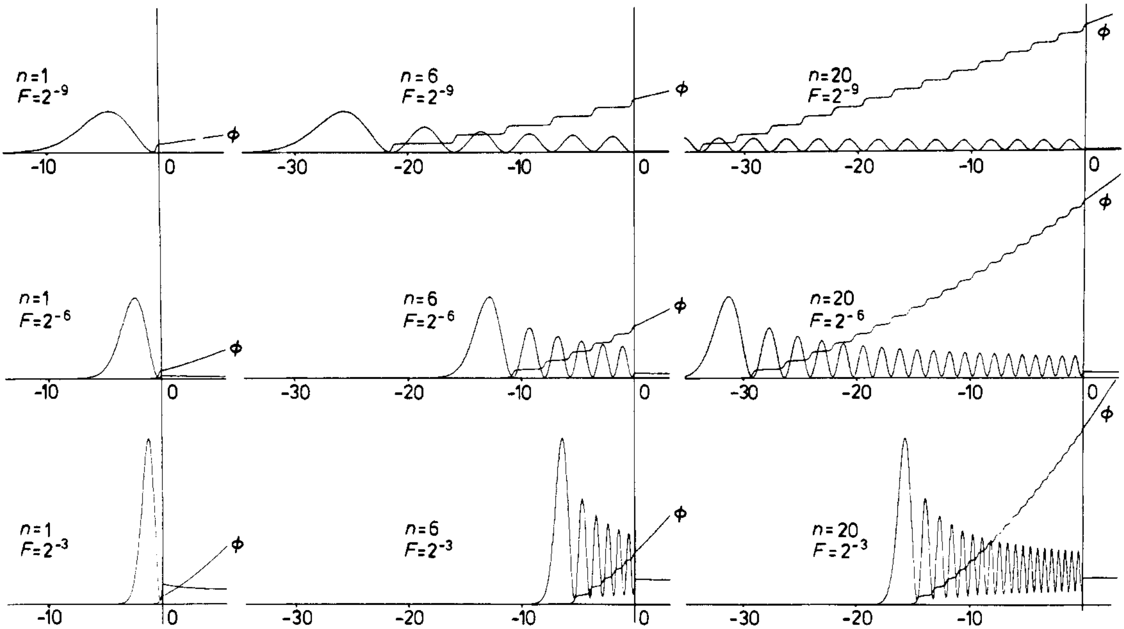

Figure 3. The modulus square and phase ϕ of the complex energy eigenfunctions, for nodal quantum numbers $n = 1, 6$ and 20 and dimensionless field intensities 2^{-9}, 2^{-6} and 2^{-3}. The distance is measured in units of $l = \hbar^2/\beta m$. As indicated also by equation (3), length scales as λ, i.e. as $F^{-1/3}$. The phase increases by π in the neighbourhood of each quasi-node. Relative normalisation is such that $\int_{-\infty}^{0} |\psi|^2\,dx$ remains constant.

curve. For convenience, we also designate the corresponding z_{-n} and ϵ_{-n} curves isonodal or adiabatic invariant. The dashed curves connect the various solutions for given $F = 4/\lambda^3$. Their individual shapes (to be discussed in § 7) are hence the same in the z and ϵ planes.

6. Approximate formulae for eigenstates near the positive real axis

From the first terms of equations (10.4.59–60) in Abramowitz and Stegun (1964) we obtain asymptotic formulae valid for $|z| \gg 1$:

$$u_0(z) \approx \pi^{-1/2} z^{-1/4} \cos \eta \quad \text{when } |\arg z| < 2\pi/3 \tag{19a}$$

$$u_+(z) = \tfrac{1}{2}\pi^{-1/2} z^{-1/4} \exp(\mathrm{i}\eta) \quad \text{when } -2\pi/3 < \arg z < 4\pi/3. \tag{19b}$$

Here $\eta = \frac{2}{3}z^{3/2} - \pi/4$, and (13) was used in (19b). With these, the eigenvalue condition (16) becomes

$$\tfrac{1}{4}z^{-1/2}(\exp(2\mathrm{i}\eta) + 1) = (2\mathrm{i}\lambda)^{-1}.$$

Setting

$$u + \mathrm{i}v \equiv \tfrac{4}{3}\mathrm{i}z^{3/2} \approx \tfrac{4}{3}\mathrm{i}x^{3/2} - 2yx^{1/2} \qquad \text{for } |y/x| \ll 1 \tag{20}$$

the eigenvalue condition becomes $\exp(u + \mathrm{i}v) + \mathrm{i} = 2z^{1/2}/\lambda \approx 2x^{1/2}/\lambda$. Real and imaginary parts give

$$e^{u} \cos v \approx 2x^{1/2}/\lambda, \qquad e^{u} \sin v \approx -1. \tag{21a, b}$$

Squaring and adding:

$$e^{2u} \approx 1 + 4x/\lambda^{2}, \tag{21c}$$

i.e.

$$-2u/4x^{1/2} \approx y \approx -(1/4x^{1/2}) \ln(1 + 4x/\lambda^{2}) \tag{22a}$$

or

$$\mathrm{Im}(\epsilon) \approx [-F/4(2\mathrm{Re}(\epsilon))^{1/2}] \ln(1 + 2\,\mathrm{Re}(\epsilon)). \tag{22b}$$

From (21*b*, *c* and *a*),

$$\sin v \approx \sin(\tfrac{4}{3}x^{3/2}) \approx -e^{-u} \approx -(1 + 4x/\lambda^{2})^{-1/2}; \qquad \cos v > 0.$$

By sketching the graphs of $\sin(\frac{4}{3}x^{3/2})$ and $-(1 + 4x/\lambda^{2})^{-1/2}$ and discarding solutions for which $\cos v < 0$, it is easy to see that if $z_n = x_n + iy_n$ is a solution, then

$$\tfrac{4}{3}x_n^{3/2} \approx 2n\pi - \delta_n$$

with $\delta_n = \sin^{-1}(1 + 4x_n/\lambda^{2})^{-1/2}$, so that

$$x_n \approx [(3n\pi/2) - \tfrac{3}{4}\sin^{-1}(1 + 4x_n/\lambda^{2})^{-1/2}]^{2/3} \tag{23a}$$

or

$$\mathrm{Re}(\epsilon_n) \approx \tfrac{1}{2}[3\{n\pi - \tfrac{1}{2}\sin^{-1}(1 + 2\,\mathrm{Re}(\epsilon_n))^{-1/2}\}F]^{2/3}. \tag{23b}$$

Taking $\delta_n = 0$ or $\delta_n = \pi/2$ in zeroth approximation in (23*a*) and iterating (this can be easily programmed even on a pocket calculator), one rapidly obtains convergence for x_n; y_n then follows from (22*a*). Table 1 compares results so obtained with results obtained by numerically solving the exact condition (16)—the latter are correct to at least twelve decimal places. On retaining next higher order terms, one obtains

$$y = 2x \tan(\tfrac{4}{3}x^{3/2} - \tfrac{1}{2}y^{2}x^{-1/2}) + \lambda x^{1/2}.$$

When solving this and (22*a*) simultaneously, and retaining only solutions for which $\cos v > 0$, one obtains further improvement in accuracy. Despite its simple derivation (22*a*, *b*) is thus remarkably accurate over a wide range of F, and will be further discussed below.

Other approximate formulae may be derived by expanding $u_0(z)u_+(z)$ in a Taylor series around c_n. For $n > 0$, the formula

$$z_n \approx c_n + \lambda^{-1} - ic_n^{1/2}\lambda^{-2} + O(\lambda^{-3}) \tag{24}$$

is everywhere accurate to at least three significant figures for all $\lambda \geqslant 10$ ($F \leqslant 0{\cdot}004$), and to at least four significant figures if also $n > 1$. The imaginary part, for such small F, is much smaller than the real part, and often comparable in size with the error in z, so that it is not given nearly as accurately by (24) as by (22). The formula

$$z_n \approx c_n - \tfrac{1}{2}ic_n^{-1/2}[-1 + (1 + 4ic_n^{1/2}/\lambda)^{1/2}] \tag{24a}$$

yields some improvement over (24).

Table 1. Eigenvalues ϵ and $z = x + iy$ near the positive real axis. Values given are accurate to twelve or more significant digits in the complex quantities ϵ and z; whenever the imaginary part (second column) is a few orders smaller than the real part, its number of significant digits is correspondingly reduced. The last two columns compare the approximate results $z_a \equiv x_a + iy_a$ obtained by the method described below equation (23*b*) with the accurate results: A value $XR = +2{\cdot}1$ indicates a relative error $(x - x_a)/x$ of $+10^{-2\cdot1}$, whereas $YR = -2{\cdot}5$ indicates a relative error $(y - y_a)/y = -10^{-2\cdot5}$. Thus the magnitudes of XR and YR give approximately the number of significant figures yielded by the approximate formulae (22*a*, *b*) and (23*a*, *b*), and the sign of XR gives the sign of the error.

N	F	ϵ		z		XR	YR
	10^{-9}	0·000001856257	−0·000000000000	2·33873741038	−0·00000061317	2·1	1·9
	10^{-6}	0·000186075964	−0·000000004877	2·34441023334	−0·00006144667	2·1	1·8
	10^{-3}	0·019054246118	−0·000048923491	2·40068457738	−0·00616397361	2·1	1·6
1	1	2·130370150393	−0·205644918777	2·68409819655	−0·25909636197	2·4	1·7
	10^{3}	225·0665064115	−72·13983705305	2·83566029054	−0·90890499239	1·9	−2·5
	10^{6}	23429·51179447	−12379·55140694	2·95193350986	−1·55972574059	1·3	−1·6
	10^{9}	2464631·980756	−1720354·634256	3·10524171280	−2·16751101698	1·0	−1·3
	10^{-9}	0·000007160769	−0·000000000001	9·02200320360	−0·00000119204	−13·8	4·5
	10^{-6}	0·000716526717	−0·000000009460	9·02767093096	−0·00011919413	−9·9	3·5
	10^{-3}	0·072081265905	−0·000088943188	9·08167042174	−0·01120613947	−6·0	2·5
6	1	7·332637993889	−0·180024918949	9·23854495976	−0·22681718489	−4·3	2·6
	10^{3}	736·6421892487	−47·53627777989	9·28111000475	−0·59891957009	3·3	3·8
	10^{6}	73795·10513137	−7745·498578316	9·29760063342	−0·97587167008	2·7	−3·1
	10^{9}	7395829·150694	−1072909·518000	9·31816082839	−1·35178128636	2·4	−2·7
	10^{-9}	0·000016300796	−0·000000000001	20·53771593024	−0·00000179847	−14·2	4·8
	10^{-6}	0·001630529090	−0·000000014255	20·54337922337	−0·00017960641	−10·2	3·8
	10^{-3}	0·163457424664	−0·000123819734	20·59434500954	−0·01560030897	−6·4	2·9
20	1	16·42245864978	−0·153697962121	20·69100134389	−0·19364729780	−5·7	3·3
	10^{3}	1643·700225831	−35·31213422923	20·70932414241	−0·44490501232	4·2	4·5
	10^{6}	164406·0656938	−5538·365661509	20·71386628980	−0·69779034790	3·7	−4·1
	10^{9}	16444203·12571	−754518·3611735	20·71839766683	−0·95063356577	3·4	−3·7

7. Semiclassical treatment of eigenvalues near the positive real axis

From (22)–(23) it follows that whenever

$$4x/\lambda^2 \ll 1, \qquad \text{i.e. } \mathrm{Re}(\epsilon) \ll \tfrac{1}{2} \qquad \text{or } (2F)^{2/3}x \ll 1, \tag{25}$$

(that is, in weak fields, when the real part of the energy is much less than the modulus of the bound state energy in a pure delta potential),

$$x_n \sim [\tfrac{3}{2}(n-\tfrac{1}{4})\pi]^{2/3} \sim c_n \tag{23c}$$

and

$$y_n \sim x_n^{1/2}/\lambda^2. \tag{22c}$$

In view of (18*a*), these also follow from (24) or (24*a*). With (6), they give

$$\tau E_n/\hbar = \epsilon_n \sim \tfrac{1}{2}[3(n-\tfrac{1}{4})\pi F]^{2/3} - \tfrac{1}{4}\mathrm{i}[3(n-\tfrac{1}{4})\pi]^{1/3}F^{4/3} \tag{26}$$

so that as $F \to 0$ (weak field) the imaginary parts of E, ϵ and z tend to zero much more rapidly than the real part, and the eigenvalues become almost real. Here, as elsewhere in the paper, we reserve the symbol $\sim$ to denote asymptotic approximations dependent on condition (25). We now show that quasi-stationary states corresponding to these eigenvalues describe a particle that oscillates between a classical turning point at position

$$s_{tp} = -\mathrm{Re}(E)/q\mathscr{E} = (-l \ \mathrm{Re}(\epsilon)/F) \approx -[3(n-\tfrac{1}{4})\pi]^{2/3} l/2F^{1/3}, \tag{27}$$

and the delta potential well at the origin ($s = 0$). With the energy becoming almost real for small F, we consider the following semiclassical argument, which is certainly valid at least when the quantum number n is large:

The velocity v of the particle incident on the delta potential at $s = 0$ follows from $\frac{1}{2}mv^2 = E \propto \epsilon \propto F^{2/3}$. With v proportional to $F^{1/3}$ tending to zero with F, and the delta discontinuity in the potential unchanged, we may reasonably expect the transmission coefficient T to tend to zero with F. An exact calculation for real E shows that when $z = x \gg 1$

$$T \underset{\gg 1}{\overset{x}{\approx}} (1+2\epsilon)^{-1} \underset{\ll 1}{\overset{\epsilon}{\sim}} 2\epsilon = (2F)^{2/3} x. \tag{28}$$

Provided that $T \ll 1$ (which is ensured by (25)), the decay probability per unit time equals the transmission probability through the delta potential well divided by the semiclassical period P; it also equals $-(2/\hbar)\,\mathrm{Im}(E) \equiv \Gamma/\hbar$. Therefore

$$\Gamma \equiv -2\,\mathrm{Im}(E) \sim \hbar T/P \sim \hbar(2F)^{2/3} x/P. \tag{29}$$

The period P of a classical particle oscillating between s_{tp} and $s = 0$ is given by

$$|s_{tp}| = \tfrac{1}{2}a(\tfrac{1}{2}P)^2,$$

with $a = q\mathscr{E}/m = Fm\beta^3/\hbar^4$ the uniform acceleration. With (27), this yields

$$\tfrac{1}{2}P = \left(\frac{-2s_{tp}}{a}\right)^{1/2} \sim \frac{[3(n-\frac{1}{4})\pi]^{1/3}\hbar^3}{F^{2/3}m\beta^2} = \frac{[3(n-\frac{1}{4})\pi]^{1/3}\tau}{F^{2/3}} \sim 2^{1/3} x_n^{1/2} \tau/F^{2/3}. \tag{30}$$

Substituting this and (23c) in (29), we obtain exactly the same $\mathrm{Im}(E_n)$ as in (26), which for $F < 10^{-10}$ is accurate to eight significant figures when $n \geqslant 6$.

From (22a) we note that for $x > \lambda^2/8$, i.e.

$$\mathrm{Re}(\epsilon) > \tfrac{1}{4}, \tag{25a}$$

$\mathrm{d}y/\mathrm{d}x > 0$, i.e. for constant F the modulus of the imaginary part of the energy decreases as n increases. (For $\mathrm{Re}(\epsilon) < \frac{1}{4}$ this is not true; in fact when (25) holds, (22c) follows.) This behaviour is also evident from figure 1.

This somewhat surprising result implies that the lifetime increases with increasing energy above the top of the delta potential well at $s = 0$! It results from the linear variation in the potential $V(s)$ in the region where the particle oscillates between S_{tp} and $s = 0$, and from the non-analytic delta 'discontinuous' behaviour of $V(s)$ at $s = 0$. The latter produces a variation of the transmission coefficient T (first part of (28)) against energy, which under condition (25a) is slower than the variation of the semiclassical period P (which is proportional to $(\mathrm{Re}(E))^{1/2}$).

We have also found (Malherbe 1976) that the effect of a potential step (or even a mere discontinuity of slope) rather than a delta potential well at $s = 0$ will produce similar results—these results with certain others for a variety of potentials are intended for publication hereafter. The mentioned property is independent of whether the potential on the right (in the outgoing wave region) is linear or constant. A semiclassical analysis similar to the above would suggest that for any potential with a discontinuity in some derivative (in the present case, the 'derivative' is of order -1; it could also be of order zero), and such that the semiclassical period increases with real energy as E^{α}, with α positive, high-energy resonant states above the discontinuity would have lifetimes increasing as E^{α} multiplied by a logarithmic factor. In particular, if in the region of semiclassical oscillation the potential $V(s)$ is proportional to some positive power less than two of the coordinate, this behaviour can be expected for states high above the discontinuity.

Indeed, Newton (1960) quotes a result for $\alpha = -\frac{1}{2}$ (corresponding to a semiclassical period *decreasing* with energy E). For high-order ($n \gg 1$) resonant states the effects of a discontinuity in the mth order derivative of the radial potential at position $r = R$ are that the poles of the Green function are at

$$E_n = \hbar^2 k^2/2m \qquad \text{with} \qquad Rk = \sim n\pi + \tfrac{1}{2}\mathrm{i}(m+2)\ln n. \tag{26a}$$

This is in striking similarity to (18a) and (22b), which like (28) shows that for real $\epsilon \ll 1$, the delta function potential well acts as an almost impenetrable barrier; although (22b) and (23b) show that to lowest order, the formula for ϵ_n is not overly much affected when T becomes comparable to 1. Now (26a) suggests, and our own exact results on various potential models confirm, that discontinuities in the potential, or some derivative of it, have an effect similar to that of a delta function well.

The states z_{-n} of (18b) have imaginary parts exceeding their real parts and far exceeding the differences between their real parts. They cannot contribute any sharp resonances, and apart from having very short lifetimes they do not appear to have a simple physical interpretation (Nussenzveig 1959).

8. Asymptotic formulae for the 'ground state' eigenvalue

As already discussed in § 4, case (a), the state obtained by applying an electric field to the ground state of a pure delta potential corresponds to z tending to $-\infty$ as F tends to zero. Hence we use the formulae (10.4.59) and (10.4.63) from Abramowitz and Stegun (1964) together with (11) and (12) in (16), and obtain

$$(\lambda\mathrm{i})^{-1} = 2\pi u_0(z)u_+(z) \approx \tfrac{1}{2}(-z)^{-1/2}\{\tfrac{1}{2}\exp(-\tfrac{4}{3}(-z)^{3/2})A^2 - \mathrm{i}AB\}$$

or

$$-z = \{\tfrac{1}{2}\lambda[AB + \tfrac{1}{2}\mathrm{i}\exp(-\tfrac{4}{3}(-z)^{3/2})A^2]\}^2 \tag{31}$$

where

$$A = \sum_0^\infty c_k[-\tfrac{4}{3}(-z)^{3/2}]^{-k} \qquad \text{and} \qquad B = \sum_0^\infty c_k[\tfrac{4}{3}(-z)^{3/2}]^{-k}$$

are asymptotic series, with c_k defined in (10.4.58) of Abramowitz and Stegun (1964).

To zeroth order in F, one sets $A = B = 1$ and ignores the exponentially small second term in (31), to obtain $-z_0 = \lambda^2/4$, $\epsilon = -\frac{1}{2}$, the ground state in a pure delta potential. Iterating with (31), one obtains

$$-4z_0 = [\lambda^2 + \mathrm{i}\lambda^2\exp(-\lambda^3/6)]\left(1 + \sum_1^\infty \mathrm{d}_n\lambda^{-3n}\right)$$

i.e.

$$\epsilon_0 = [-\tfrac{1}{2} - \tfrac{1}{2}\mathrm{i}\exp(-2/3F)]\left[1 + \sum_1^\infty \mathrm{d}_n(F/4)^n\right]. \tag{32}$$

Ignoring for the moment the last factor in (32) (an asymptotic series), we note a striking similarity between $\mathrm{Im}(\epsilon_0)$ and the approximate formula $w = 4F^{-1}\exp(-2/3F)$ for the decay rate of a hydrogen atom in an electric field (Landau and Lifschitz 1965). In view of the already noted analogy between the three-dimensional hydrogen atom and the one-dimensional delta potential (Herrick and Stillinger 1974), this is perhaps not surprising.

The result (32) was obtained in 1970 by Brown (1978, private communication) who investigated whether Rayleigh–Schrödinger perturbation theory gave the correct series expansion for $\mathrm{Re}(\epsilon_0)$, and compared the perturbation series with (32).

Great care is, however, needed in handling asymptotic series. For instance, on using for $u_+(x)$ the perfectly valid asymptotic series of which (19*b*) is the first term together with the same series for $u_0(z)$ as was used in (31), one obtains an equation for z quite similar to (31), but without an i appearing anywhere, and with all coefficients of asymptotic series real! Numerically or analytically iterating this equation gives $\mathrm{Re}(\epsilon_0)$ to good accuracy when $|z| \gg 1$, i.e. asymptotically, but also gives $\mathrm{Im}(\epsilon_0) \equiv 0$! Since *asymptotically* $\mathrm{Im}(\epsilon_0)$ is zero, this is not very surprising. To check these points, and particularly the accuracy of an iterative solution of (31), we summed the convergent Taylor series for the Airy functions u_0 and u_+ in extended precision (approximately thirty three decimal digits carried) in order that the error in z (due to cancellation of terms) be smaller in magnitude than $\mathrm{Im}\, z$.

Table 2 compares the result of iterating (31) numerically (which we believe is the same as using (32)) with the correct numerical result using convergent series. The last is accurate everywhere to at least twelve significant figures, and from the comparison it appears that for $F \leqslant 2^{-6}$, (32) is too.

Table 2. Complex eigenvalues ϵ for the state obtained by applying an electric field to the bound state of a delta function potential. *XR* and *YR* have the same meaning as in table 1, but with approximate values now obtained from equation (32). *YRR* has a meaning similar to *YR*, with approximate value obtained from the first term of equation (32).

F		ϵ	*XR*	*YR*	*YRR*
2^{-8}	−0·500009538345	$-3{\cdot}7719037579\times10^{-75}$	>12	<−12	−2·2
2^{-7}	−0·500038172660	$-4{\cdot}2998151091\times10^{-38}$	>12	<−12	−1·9
2^{-6}	−0·500153002629	$-1{\cdot}4370430689\times10^{-19}$	>12	<−12	−1·6
2^{-5}	−0·500617255201	$-2{\cdot}5709647509\times10^{-10}$	9·9	−9·2	−1·3
2^{-4}	−0·502578354630	$-1{\cdot}0332591239\times10^{-5}$	6·4	−3·7	−0·89
2^{-3}	−0·511571887421	$-1{\cdot}8078998100\times10^{-3}$	2·8	−0·9	−0·4
2^{-2}	−0·536883147240	$-2{\cdot}2265005463\times10^{-2}$	1·5	−0·1	
2^{-1}	−0·573818709131	$-9{\cdot}4800178749\times10^{-2}$			
1	−0·607216002651	−0·26458169159			
10	−0·202405226365	−2·9400778226			
10^2	+5·42988029386	−20·122841005			
10^3	45·3021549684	−118·41399756			
10^4	283·395118950	−650·961082315			
10^5	1603·60050606	−3446·4699797			

Acknowledgments

The authors are much indebted to Professors W E Frahn, R E Peierls and W Byers Brown for constructive criticism.

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